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Scattering fidelity in elastodynamics

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The recent introduction of the concept of scattering fidelity causes us to revisit the experiment by Lobkis and Weaver [Phys. Rev. Lett. **90**, 254302 (2003)]. There, the "distortion" of the coda of an acoustic signal is measured under temperature changes. This quantity is, in fact, the negative logarithm of scattering fidelity. We reanalyze their experimental data for two samples, and we find good agreement with random matrix predictions for the standard fidelity. Usually, one may expect such an agreement for chaotic systems, only. While the first sample may indeed be assumed chaotic, for the second sample, a perfect cuboid, such an agreement is surprising. For the first sample, the random matrix analysis yields perturbation strengths compatible with semiclassical predictions. For the cuboid, the measured perturbation strengths are by a common factor of $\frac{5}{3}$ too large. Apart from that, the experimental curves for the distortion are well reproduced.

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Lobkis and Weaver (henceforth LW) have measured the sensitivity of elastic coda waves to temperature changes [1]. For this purpose they used the cross-correlation function between these waves at different temperatures. The use of coda waves, i.e., of the response after some initial transient, is essential in getting rid of system specific short time effects and to see generic features. After correcting for a trivial term due to a change of volume and wave speeds they quantify the remaining changes as "distortion," and study its behavior as a function of time. We shall show that this quantity is the negative logarithm of the "scattering fidelity" introduced and measured in [2,3]. The change in temperature provides the "perturbation" of the dynamics characteristic of all experiments in echo dynamics. For sufficiently chaotic dynamics in systems weakly coupled to decay channels, the scattering fidelity in turn approaches the standard fidelity amplitude. Fidelity, the absolute value squared of the fidelity amplitude, has received a great deal of attention in recent years, as it is used as a benchmark in quantum information processes [4], and in the context of quantum chaos (for a partial overview and pertinent references, see [5]).

A random matrix description of fidelity decay [6] has explained comparable experiments with chaotic microwave cavities [2,3] quite well. Random matrix theory (RMT) makes a definite statement on the form of the fidelity amplitude as a function of time. It yields a unified description of the "perturbative" and the "Fermi golden rule" [7–9] regime. We expect this analysis to provide a more appropriate interpretation of the experiments than the original treatment in [1]. RMT still contains the perturbation strength as a free parameter, which must be determined independently. For quantum chaotic systems, different semiclassical methods can be used for that purpose [2,9,10]. In the present case, we may still use the result of Ref. [1].

We analyzed the data for two three-dimensional samples, measured by LW. The first object is the so-called "medium block" (after extra cut) which is supposed to have dominantly chaotic dynamics, with no symmetries left. The second object is a cuboid, called "rectangle," where the dynamics is not chaotic, but due to mode conversion it is also not integrable and actually known to display random matrix behavior as far as spectral statistics are concerned [11]. In both cases, we obtained very good agreement between experiment and the RMT prediction, if we fit the strength of perturbation. For the first sample, the fitted strength is in reasonable agreement with the theoretical result of LW. For the second sample, we obtain values that are larger by a common factor of $\frac{5}{3}$, approximately.

The second case is particularly important as the behavior of such systems is not well understood. Both the general agreement of the shape of the fidelity decay with the RMT prediction as well as the failure of the semiclassical estimate for the perturbation strength are lacking a theoretical explanation. The concepts of fidelity and scattering fidelity together with the RMT approach set an appropriate frame, where these questions can be discussed.

Scattering fidelity. The scattering fidelity [2] is defined as

$$f_{ab}(t) = \langle \hat{S}_{ab}(t)^* \hat{S}'_{ab}(t) \rangle / \sqrt{\langle |\hat{S}_{ab}(t)|^2 \rangle \langle |\hat{S}'_{ab}(t)|^2 \rangle}. \qquad (1)$$

Here, $\langle \hat{S}_{ab}(t)^* \hat{S}'_{ab}(t) \rangle$ is the Fourier transform of the cross-correlation function of a scattering matrix element for the unperturbed and perturbed system, respectively. Similarly, $\langle |\hat{S}_{ab}(t)|^2 \rangle$ and $\langle |\hat{S}'_{ab}(t)|^2 \rangle$ are the Fourier transforms of the corresponding autocorrelation functions, used for proper normalization. It can be shown that appropriate averaging (denoted by $\langle \cdots \rangle$) yields the standard fidelity amplitude $f(t)$ for chaotic systems weakly coupled to the scattering channels [2]. Using the linear response result wrapped in an exponential [6], we obtain

$$-\ln f(t) = \lambda_0^2\left[\frac{x}{2} + x^2 - \int_0^x dx' \int_0^{x'} dx'' b_2(x'')\right]. \qquad (2)$$

Here, $\lambda_0/(2\pi)$ is the average size of an off-diagonal element of the perturbation in units of the mean level spacing in the unperturbed system. The variable x measures time in units of the Heisenberg time, $x=t/t_H$, and $b_2(x)$ is the two point form factor describing the spectral correlations of the unperturbed Hamiltonian, which is taken from the Gaussian orthogonal ensemble. The approximate result is dominated for short times $t<t_H$ by the linear term, and for long times $t>t_H$ by the quadratic term. There, the double integral just compensates the linear term up to a remainder, logarithmic in x. For the fidelity amplitude in turn, this means that for large $\lambda_0 \ll 1$ the decay is nearly exponential (Fermi golden rule regime), while for small $\lambda_0 \ll 1$ it is nearly Gaussian (perturbative regime) [6–9]. Note that this distinction is irrelevant for the distortion, where the perturbation strength appears as a common prefactor.

In a microwave experiment excellent agreement between $f_{ab}(t)$ and $f(t)$, as given by Eq. (2), has been found [2,3]. For closed chaotic systems the validity of this approximation has been demonstrated numerically in various expamples such as the kicked rotor [12].

In Ref. [1] the authors measure the acoustic response to a short piezoelectric pulse as a function of time. They consider the normalized cross correlation between two such signals obtained at the temperatures T_1 and T_2:

$$X(\varepsilon) = \frac{\int dt\, S_{T_1}(t) S_{T_2}(t(1+\varepsilon))}{\sqrt{\int dt\, S_{T_1}^2(t) \int dt\, S_{T_2}^2(t(1+\varepsilon))}}. \qquad (3)$$

This expression displays a structure similar to Eq. (1). The time averaging over a small window, performed in Eq. (3), corresponds to a smoothing of the correlation functions in Eq. (1). The selection of ε, such that the correlation function $X(\varepsilon)=X_{\max}$ becomes maximal, eliminates the trivial effects due to dilation and change of wavespeed, caused by the temperature change. This is equivalent to the spectral unfolding performed in Refs. [2,3] to eliminate the volume effect of a moving wall. The distortion is defined as $D(t)=-\ln(X_{\max})$, where the time dependence is given by the "age" of the signal, i.e., the center of the small time interval over which the correlation function $X(\varepsilon)$ was evaluated. If formulated as a scattering process, we find

$$D(t) = -\ln[f_{aa}(t)] = -\ln f(t), \qquad (4)$$

where the scattering channel a is defined by the transducer, that transmits excitation to and from the sample. Thus, for sufficiently chaotic samples in the elastodynamic scattering experiments, we expect the scattering fidelity to be equal to the fidelity amplitude, and the latter to be well described by the RMT result, Eq. (2).

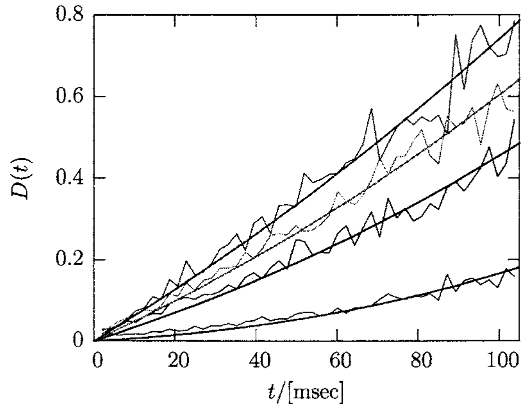

FIG. 1. The distortions for the medium block as plotted in Fig. 4 of Ref. [1]. The thin jagged lines correspond to measurements in the frequency ranges 300, 600, 700, and 800 kHz (from bottom to top). The thick lines show the corresponding theoretical curves according to the Eqs. (2) and (4), using the values of λ_0 given in Table I (full RMT fit).

LW describe the behavior of $D(t)$ with the help of a ray picture of the resonating acoustic waves, the assumption of random reflection angles along the ray paths, and an estimate for the mode conversion rates between dilational and shear waves. Similar to the Fermi golden rule regime, they obtain a linear time dependence:

$$D(t) = \lambda_1^2 t/(2t_H), \quad \lambda_1 = \nu\sqrt{2t_H C\Delta^2 V/S}, \qquad (5)$$

where $C\approx 3.26\times 10^{-10}/(\mathrm{K}^2\,\mathrm{cm\,msec}^{-1})$ denotes the distortion coefficient (see Ref. [1]), Δ denotes the temperature difference T_1-T_2 in Kelvin, while V and S denote the volume and the surface of the sample. Since the RMT result has the same behavior at asymptotically small times, we may set $\lambda_0=\lambda_1$, to obtain a parameter-free theory for distortion, valid also at large times. From Refs. [13,14], we obtain the following expression for the Heisenberg time t_H:

$$t_H = \frac{4\pi}{c_p^3}V(2q^3+1)\nu^2 + \frac{\pi}{2\,c_p^2}S\left(3q + \frac{2}{q^2-1}\right)\nu. \qquad (6)$$

Here, $c_p=637$ cm/msec and $c_s=c_p/q=316$ cm/msec are the longitudinal and the transverse (shear) wave velocities. The frequency (range) is denoted by ν.

Medium block. For that sample, $V=906\,\mathrm{cm}^3$ and $S=636\,\mathrm{cm}^2$, and the temperature difference was $\Delta=4$ K. The distortion, measured as a function of time is shown in Fig. 1. With the Heisenberg times given in Eq. (6), we have fit the perturbation strength λ_0 and the results are given as "full RMT fit" in Table I. For experimental reasons, the data for $D(t)$ are reliable for $t>20$ msec. The fits have been restricted correspondingly.

The agreement between the measured distortions or scattering fidelities and RMT is within the statistical error of the data. In most cases deviations from linear behavior are not noticable to the eye, except for the 300 kHz data, where $t_H \approx 78.4$ msec [15] lies within the time range of the figure.

TABLE I. Medium block: Table of linear fit and full RMT fit values for the dimensionless perturbation strength λ_0 and the respective Heisenberg times for the four frequency ranges.

ν(kHz)	λ_0	(Lin. fit)	(Full fit)	t_H(msec)
300		0.471	0.298	78.4
600		1.584	1.381	294.6
700		2.136	1.929	397.3
800		2.709	2.505	515.3

However, the influence of the quadratic term on the fit values for λ_0 is large. In Table I, they are contrasted to those obtained by a linear fit in the spirit of LW.

In Fig. 2, the perturbation strengths λ_0 obtained from the linear fit (diamonds) and the full fit (circles) are compared to the estimate from Eq. (5). The difference between both fitting methods is clearly noticeable. The full RMT analysis yields values for λ_0 quite close to the semiclassical prediction of LW. We have no explanation for the remaining deviations. It is not clear, whether these are statistically acceptable, whether chaoticity is not perfect, or whether there is some other reason. We may recall that in Ref. [2], the experimental perturbation strength did also differ from the theoretical estimate, as it was not performed sufficiently far in the semiclassical regime.

Rectangle. In Figs. 3 and 4, we analyze data for the rectangle, a perfect cuboid, in a similar way as above. Clearly, a scalar wave equation would lead to integrable ray dynamics, where our RMT model must fail. However, in the present case, the wave field has two components (dilatational and shear waves), which are coupled due to mode conversion. The corresponding classical dynamics is marginally stable, but may still be ergodic. Recently, the spectral statistics of such an elastodynamic system has been studied thoroughly [11]. Apart from certain symmetries, the statistical measures accurately show RMT behavior.

From a theoretical point of view, it remains an open question, whether scattering fidelity still agrees with standard fidelity, and whether the behavior of the fidelity amplitude can be described by RMT. The following analysis with Eq. (2) will shed some light on these questions.

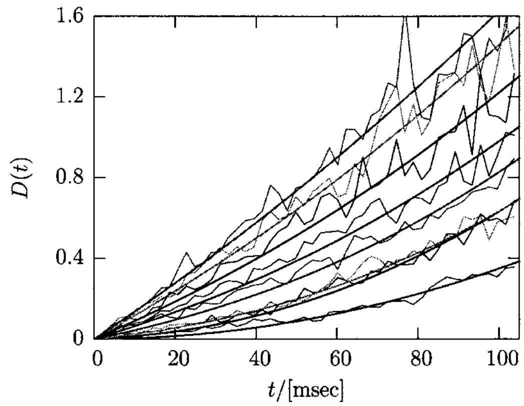

FIG. 3. The distortion as a function of time, for the rectangle. The thin jagged lines correspond to measurements in the frequency ranges from 100 to 800 kHz, in steps of 100 kHz (from bottom to top). The thick smooth lines show the best theoretical fits, according to Eqs. (2) and (4), with λ_0 given in Table II (full RMT fit).

For the rectangle, we have data for eight frequency windows at ν=100 kHz, 200 kHz,…, 800 kHz. The corresponding Heisenberg times range from 10 to 500 msec (see Table II). In Fig. 3, one can see a transition from a linear to a quadratic decay characteristic of the RMT expression, Eq. (2). Here, it is really surprising that the RMT expression describes the data so well. The perturbation strengths obtained from fits to Eq. (2) are plotted in Fig. 4 (circles). Except for a constant factor of $\frac{5}{3}$, approximately, the result follows the theoretical expectation [Eq. (5), solid line]. This is demonstrated by a fit for the distortion coefficient on the basis of Eq. (5), which yields $C=8.90\times 10^{-10}/(\mathrm{K}^2\,\mathrm{cm\,msec}^{-1})$ (dashed line). Considering that the argument in LW leading to Eq. (5) is based on fast chaotic ray mixing this difference is less surprising, than the agreement in scaling.

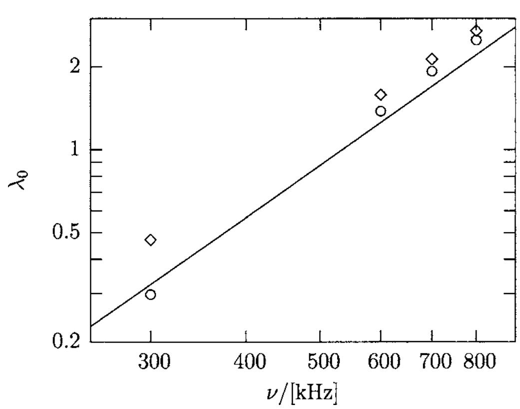

FIG. 2. Medium block: The perturbation strength as a function of the frequency range. The circles show the values for λ_0 obtained from a fit with Eq. (2), while the diamonds show the corresponding values when only the linear term of Eq. (2) is taken into account. The solid line gives the perturbation strength as obtained from Eq. (5).

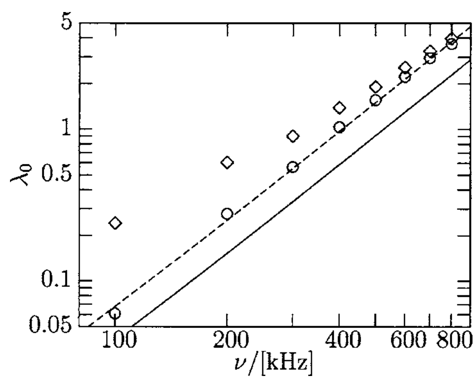

FIG. 4. Rectangle: The perturbation strength as a function of the frequency range. The circles show the values for λ_0 as obtained from Eq. (2), and from the same expression, taking into account only the linear term (diamonds). The solid line gives the perturbation strength as given by Eq. (5). The dashed line shows the same expression, but with the distortion coefficient fitted to the data points (circles; see text).

TABLE II. Rectangle: Table of linear and full RMT fit values for the dimensionless perturbation strength λ_0 together with the respective Heisenberg times.

ν (kHz)	λ_0 (Lin. fit)	(Full fit)	t_H (msec)
100	0.241	0.061	10.36
200	0.607	0.279	35.75
300	0.903	0.567	76.19
400	1.389	1.033	131.68
500	1.905	1.561	202.21
600	2.564	2.227	287.78
700	3.282	2.957	388.39
800	3.969	3.662	504.04

In this paper, we identified a previously published elastomechanic scattering experiment as an experiment that measures scattering fidelity in a setting usually called echodynamics. As in the case of electromagnetic billiards, it turns out that a simple random matrix model describes the decay of the scattering fidelity very well, despite the fact that these are much more complicated systems. The RMT model leads to an additional quadratic term, which is the most important difference to the model used in [1]. For the first sample with suposedly chaotic ray dynamics, it leads to a better (though not perfect) parameter-free description of the measured distortion.

For the rectangle, the RMT description is still valid, though the perturbation strengths are larger than predicted by LW's analysis. The applicability of the RMT approach shows that at least in one important aspect not only energies but also wave functions behave like those of a chaotic system. In this sense, our results complement those of [11]. Further studies of this situation will be of great interest.

The high quality factors of elastomechanic experiments, as well as the possibility to measure explicitly in the time domain, make these experiments particularly welcome. Among possible experiments, we believe that it will be worthwhile to analyze strong perturbation data, e.g., larger temperature differences. Such experiments are more difficult, due to precision problems, but may serve to explore the limits of RMT, or detect other regimes of fidelity decay. This is particularly interesting, because the exact solution of the RMT model is now available [16], and it shows a small local maximum of fidelity at the Heisenberg time.

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